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Note

Connectivity of iterated line graphs

Yehong Shao *

Arts and Science, Ohio University Southern, Ironton, OH 45638, United States

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ABSTRACT

Let $k \geq 0$ be an integer and $L^k(G)$ be the kth iterated line graph of a graph G. Niepel and Knor proved that if G is a 4-connected graph, then $\kappa(L^2(G)) \geq 4\delta(G) - 6$. We show that the connectivity of G can be relaxed. In fact, we prove in this note that if G is an essentially 4-edge-connected and 3-connected graph, then $\kappa(L^2(G)) \geq 4\delta(G) - 6$. Similar bounds are obtained for essentially 4-edge-connected and 2-connected (1-connected) graphs.

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1. Introduction

We use [1] for terminology and notation not defined here, and consider finite, simple and connected graphs only. Let G be a graph. By $d_G(v)$ we denote the *degree* of the vertex v in G. By $\kappa(G)$ we denote the *vertex connectivity* of G. A graph is *trivial* if it contains no edges. Let X, Y be the vertex subsets of G. By $[X, Y]$ we denote the set of edges with one end in X and the other end in Y. A *cycle* of length k is denoted by C_k. An edge cut Y of G is *essential* if $G - Y$ has at least two nontrivial components. For an integer $k > 0$, a graph G is *essentially k-edge-connected* if G does not have an essential edge cut Y with $|Y| < k$. We use $\kappa'_e(G)$ to denote the *essential edge connectivity* of a graph G. For a vertex $v \in V(G)$, define

$$N_G(v) = \{u \in V(G) : u \text{ is adjacent to } v \text{ in } G\}$$

and

$$E_G(v) = \{e \in E(G) : e \text{ is incident with } v \text{ in } G\}.$$

Suppose that S is a nonempty subset of $V(G)$. A *subgraph induced* by S is a subgraph H of G such that $V(H) = S$ and $E(H)$ consists of all edges of G whose endpoints belong to S. By $G[S]$ we denote a subgraph induced by S. By $G - S$ we denote a graph obtained from G by deleting all vertices of S and their incident edges in G.

The *line graph* of a graph G, denoted by $L(G)$ or $L^1(G)$, has $E(G)$ as its vertex set, where two vertices in $L(G)$ are adjacent if and only if the corresponding edges in G have a common vertex. Then $L^k(G)$, $k = 2, 3, \ldots$, defined recursively via $L^{k+1}(G) = L(L^k(G))$ are the iterated line graphs of G.

In 1969, Chartrand and Stewart studied the edge and vertex connectivity relations between a graph G and its line graph $L(G)$ in [2]. In 2003, Knor and Niepel studied the relation between minimum degree of a graph G and the vertex connectivity of its second-iterated line graph $L^2(G)$. The following are two of their results from [3].

Theorem 1.1 (*Knor and Niepel, [3]*)**.** *Let G be a connected graph with $\delta(G) \geq 3$. Then $\kappa(L^2(G)) \geq \delta(G) - 1$.*

Theorem 1.2 (*Knor and Niepel, [3]*)**.** *Let G be a graph with $\kappa(G) \geq 4$. Then $\kappa(L^2(G)) \geq 4\delta(G) - 6$.*

* Fax: +1 740 533 4590.
E-mail address: shaoy@ohio.edu.

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The line graph of $K_{1,3}$ is a 3-cycle. The iterated line graph of a cycle is always a cycle, and the lth iterated line graph of a path of length l is K_1. Throughout this note, we always assume that G is a simple connected graph that is not a path, cycle or $K_{1,3}$. In this paper, we shall prove the following theorems. Note that Theorem 1.4(i) generalizes Theorem 1.2.

Theorem 1.3. *Let G be a graph that is not a path, cycle, or $K_{1,3}$. If $\kappa'_e(G) \geq 4$, then each of the following holds and each bound is sharp.*

$$\kappa(L^2(G)) \geq \begin{cases} 4 : & \text{if } 1 \leq \delta(G) \leq 2 \\ 6 = 4\delta(G) - 6 : & \text{if } \delta(G) = 3 \\ 4\delta(G) - 8 : & \text{if } 4 \leq \delta(G) \leq 5 \\ 14 = 4\delta(G) - 10 : & \text{if } \delta(G) = 6 \\ 16 = 4\delta(G) - 12 : & \text{if } \delta(G) = 7 \\ 4\delta(G) - 16 : & \text{if } \delta(G) \geq 8. \end{cases}$$

Note that if G is $\delta(G)$-regular, then $L^2(G)$ is $(4\delta(G) - 6)$-regular, so that $4\delta(G) - 6$ is an upper bound for the connectivity of $L^2(G)$.

Theorem 1.4. *Let G be a graph that is not a path, cycle, or $K_{1,3}$. If $\kappa'_e(G) \geq 4$, then each of the following holds and each bound is sharp.*

(i) *If $\kappa(G) \geq 3$, then $\kappa(L^2(G)) \geq 4\delta(G) - 6$.*
(ii) *If $\kappa(G) \geq 2$, then $\kappa(L^2(G)) \geq 4\delta(G) - 10$.*
(iii) *If $\kappa(G) \geq 1$, then $\kappa(L^2(G)) \geq 4\delta(G) - 16$.*

2. Lemmas

Proposition 2.1. *Let x and $d - x$ be positive integers.*

(i) *Let $g(x) = x(d - x)$. Then $g(x)$ is nondecreasing on $[0, d/2]$.*
(ii) *If $0 \leq k \leq x \leq d/2$, then $x(d - x) \geq k(d - k)$. In particular, $g(x) \geq d - 1$.*

Proof. (i) Since $g'(x) = d - 2x \geq 0$ for $0 \leq x \leq d/2$, $g(x)$ is nondecreasing on $[0, d/2]$.

(ii) It follows from (i) that $x(d-x) \geq k(d-k)$ since $0 \leq k \leq x \leq d/2$. As x, $d-x$ are positive integers, either $1 \leq x \leq d/2$ or $1 \leq d - x \leq d/2$. Then by setting $k = 1$, we have that $g(x) = x(d - x) \geq d - 1$. □

Let G be a graph. For each edge $e \in E(G)$, let $v_e \in V(L(G))$ denote its corresponding vertex in the line graph $L(G)$. Let X be a minimal essential edge cut of $L(G)$, and L_1, L_2 be two nontrivial components of $L(G) - X$. Let $f : E(G) \mapsto \{1, 2\}$ be a 2-edge-coloring of G such that $f(e) = i$ if and only if $v_e \in V(L_i)$ for $i = 1, 2$. Then Proposition 2.2 is straightforward.

Proposition 2.2. *Let G be a graph, $e_1, e_2 \in E(G)$ and $v_{e_1}, v_{e_2} \in V(L(G))$ be their corresponding vertices in the line graph $L(G)$. Then each of the following holds:*

(i) *For $i = 1, 2$, $|\{e \in E(G) : f(e) = i\}| = |V(L_i)| \geq 2$.*
(ii) *$v_{e_1}v_{e_2} \in X$ if and only if e_1, e_2 share a common vertex and $f(e_1) \neq f(e_2)$.*

A vertex of a graph G is *mono-colored* if all incident edges have the same color in G.

Proposition 2.3. *Let G be a graph, $V_{12} = \{v \in V(G) :$ there exist $e_1, e_2 \in E_G(v)$ such that $f(e_1) = 1$ and $f(e_2) = 2\}$. Then each of the following holds.*

(i) *$d_G(v) \geq 2$ for each $v \in V_{12}$.*
(ii) *Each vertex of $G - V_{12}$ is mono-colored in G, and moreover, for each component H of $G - V_{12}$, all edges with at least one end in H have the same color as the edges of H.*
(iii) *If $1 \leq |V_{12}| \leq 3$ and V_{12} is not a vertex cut, then the subgraph of G induced by V_{12}, $G[V_{12}]$, is connected.*
(iv) *If $|V_{12}| = 1$ or 2, then V_{12} is a vertex cut.*

Proof. (i) and (ii) follow immediately from the definition of V_{12}.

(iii) Since V_{12} is not a vertex cut, $G - V_{12}$ is connected. Suppose that $G[V_{12}]$ is disconnected. By $|V_{12}| \leq 3$, $G[V_{12}]$ has a component with exactly one vertex. Without loss of generality, we assume that $v_1 \in V_{12}$, which is a single vertex component of $G[V_{12}]$. By the definition of V_{12}, there exist $u_1, u_2 \in V(G - V_{12}) \cap N_G(v_1)$ such that $\{f(u_1v_1), f(u_2v_1)\} = \{1, 2\}$. Then both u_1 and u_2 are mono-colored vertices. Since V_{12} is not a vertex cut, there is a u_1–u_2 path in $G - V_{12}$, so that $G - V_{12}$ has a non-mono-colored vertex, contrary to Proposition 2.3(ii).

(iv) By way of contradiction we assume that $G - V_{12}$ is connected. If $|V_{12}| = 1$, then let $V_{12} = \{v\}$. Then there exist $u_1, u_2 \in V(G - \{v\}) \cap N_G(v)$ such that $\{f(u_1v), f(u_2v)\} = \{1, 2\}$, contrary to Proposition 2.3(ii).

If $|V_{12}| = 2$, then let $V_{12} = \{v_1, v_2\}$. If $|V(G - V_{12})| = 1$, then G is C_3 or a path of length of 2, contrary to the assumptions of this note. Thus, $|V(G - V_{12})| \geq 2$. By Proposition 2.3(iii), $G[V_{12}]$ is connected, so that $v_1v_2 \in E(G)$. Without loss of generality we may assume that $f(v_1v_2) = 2$. Then by Proposition 2.3(ii), $f(e) = 1$ for any $e \in E(G) - \{v_1v_2\}$. Then $|\{e \in E(G) : f(e)\} = 2| = 1$, contrary to Proposition 2.2(i). Thus (iv) holds. □

3. Proofs of main results

In this section, we prove Theorems 1.3 and 1.4. We follow the notations and definitions introduced in Section 2. Let G be a graph that is not a path, cycle, or $K_{1,3}$, and $\kappa'_e(G) \geq 4$. Let $u \in V(G)$ and $E_i(u) = \{e \in E_G(u) : f(e) = i\}$ for $i = 1, 2$. A vertex cut Y of a graph G is *nontrivial (trivial)* if at least two components of $G - Y$ have more than one vertex (at most one component of $G - Y$ has more than one vertex).

We first prove Proposition 3.1, which will be repeatedly used in the following claims.

Proposition 3.1. *Let $t \geq 0$ be an integer and G be a graph. If G is not isomorphic to $K_{1,t}$ and $\kappa'_e(G) \geq 4$, then the degree sum of any two adjacent vertices is at least 6.*

Proof. Suppose that there exist two adjacent vertices $u, v \in V(G)$ such that $d_G(u) + d_G(v) \leq 5$. First we assume that one of them, say u, has degree one in G. Then $d_G(v) \leq 4$. Since G is not isomorphic to $K_{1,t}$, $G - \{v\}$ has at least one edge. So $E_G(v) - \{uv\}$ is an essential edge cut of size at most 3 in G, contrary to $\kappa'_e(G) \geq 4$. Next we assume that both of $\{u, v\}$ have degree at least 2. Observe that by our assumptions now either both u and v have degree 2 or one of them has degree 3 and the other one has degree 2. If $G - \{u, v\}$ has at least one edge, then $E_G(u) \cup E_G(v) - \{uv\}$ is an essential edge cut of G of size at most 3, contrary to $\kappa'_e(G) \geq 4$. So we assume that $V(G - \{u, v\})$ is an independent vertex set of G. Since G is neither a path nor a 3-cycle, one of u and v has degree 3, while the other one has degree 2. Assume that the degree of u is 3 and the degree of v is 2. Then u is adjacent to a pendant vertex, say w, and $E_G(u) - \{uw\}$ is an essential edge cut of size 2, a contradiction. Hence Proposition 3.1 is established. □

Let X be a minimal essential edge cut of $L(G)$. If G is isomorphic to $K_{1,t}$, then $\delta(G) = 1$ and $L(G)$ is isomorphic to K_t. Clearly $|X| \geq 4\delta(G) - 6$. So we assume that G is not isomorphic to $K_{1,t}$ in the following claims.

By Propositions 2.2(ii) and 2.1(ii),

$$|X| = \sum_{u \in V_{12}} |E_1(u)|\,|E_2(u)| \geq |V_{12}|(d_G(u) - 1). \tag{1}$$

By Proposition 2.3(iv), if V_{12} is not a vertex cut, then $|V_{12}| \geq 3$. So all possible cases are considered in the following claims. In Claim 1, $|V_{12}| \geq 4$; in Claim 2, $|V_{12}| = 3$ and V_{12} is not a vertex cut; in Claim 3, $|V_{12}| \leq 3$ and V_{12} is a trivial vertex cut; in Claims 4 through 6, V_{12} is a nontrivial vertex cut, and $|V_{12}| = 3, 2, 1$ respectively.

Claim 1. *If $|V_{12}| \geq 4$, then $|X| \geq 4\delta(G) - 4$. In particular,*

$$|X| \geq \begin{cases} 4 \geq 4\delta(G) - 4 : \textit{if } 1 \leq \delta(G) \leq 2 \\ \quad 4\delta(G) - 4 : \textit{if } \delta(G) \geq 3. \end{cases}$$

Proof of Claim 1. From (1) and Proposition 2.3(i), $|X| \geq 4(\max\{2, \delta(G)\} - 1)$ and so Claim 1 holds. □

Claim 2. *If $|V_{12}| = 3$ and V_{12} is not a vertex cut, then $|X| \geq 4\delta(G) - 6$. In particular,*

$$|X| \geq \begin{cases} 4 > 4\delta(G) - 6 : \textit{if } 1 \leq \delta(G) \leq 2 \\ \quad 4\delta(G) - 6 : \textit{if } \delta(G) \geq 3. \end{cases}$$

Proof of Claim 2. Since G is a connected graph distinct from a path and 3-cycle, we have $|V(G)| \geq 4$. First we assume that $G - V_{12}$ is a connected component with exactly one vertex. Then $|V(G)| = 4$. Since $\kappa'_e(G) \geq 4$, G must be isomorphic to K_4 as any pair of independent edges must be connected by at least four edges. (Recall that $G \neq K_{1,t}$.) So by (1) and $\delta(K_4) = 3$, $|X| \geq 3(\delta(G) - 1) = 4\delta(G) - 6$.

Next we assume that $G' = G - V_{12}$ is a connected component with at least two vertices. Let $V_{12} = \{v_1, v_2, v_3\}$. By Proposition 2.3(iii), we may assume that $v_1v_2, v_2v_3 \in E(G)$. By Proposition 2.3(ii), without loss of generality, we may assume that $f(e) = 1$ for each edge $e \in [V(G'), V_{12}]$. If $v_1v_3 \notin E(G)$, then $f(v_1v_2) = f(v_2v_3) = 2$ by the definition of V_{12}; if $v_1v_3 \in E(G)$, then at least two of $\{v_1v_2, v_2v_3, v_1v_3\}$ have color 2. So we may still assume that $f(v_1v_2) = f(v_2v_3) = 2$.

Since $|E_2(v_2)| = 2$, by the first equality of (1) and Proposition 2.1(ii), we have that $|X| \geq 2(d_G(v_2) - 2) + (d_G(v_1) - 1) + (d_G(v_3) - 1) \geq 2 \cdot (\max\{3, \delta(G)\} - 2) + 2 \cdot (\max\{2, \delta(G)\} - 1)$. So Claim 2 holds and the lower bound can be obtained when $d_G(v_1) = d_G(v_2) = d_G(v_3) = \delta(G) \geq 3$. □

Claim 3. *If V_{12} is a trivial vertex cut of size at most 3, then $|X| \geq 4\delta(G) - 6$. In particular,*

$$|X| \geq \begin{cases} 5 > 4\delta(G) - 6 : \textit{if } 1 \leq \delta(G) \leq 2 \\ \quad 4\delta(G) - 6 : \textit{if } \delta(G) \geq 3. \end{cases}$$

Proof of Claim 3. Since V_{12} is a vertex cut, $G - V_{12}$ has at least two components; and since at most one component of $G - V_{12}$ is nontrivial, at least one component is a single vertex, say v. Then $\delta(G) \leq d_G(v) \leq 3$, which is implied by $|V_{12}| \leq 3$.

If $|V_{12}| = 1$, then $\delta(G) = d_G(v) = 1$. Let $V_{12} = \{v_1\}$. Since G is not isomorphic to $K_{1,t}$, at least one of the components of $G - V_{12}$ is nontrivial, say H, then $|[v_1, V(H)]| \geq \kappa'_e(G) \geq 4$. By Proposition 2.3(ii), we may assume that every edge in $[v_1, V(H)]$ has color 1. By Proposition 2.2(i), $|E_1(v_1)| \geq 2$. So $d_G(v_1) = |E_1(v_1)| + |E_2(v_1)| \geq |[v_1, V(H)]| + |E_2(v_1)| \geq 6$. By (1), $|X| \geq |E_1(v_1)|\,|E_2(v_1)| \geq 2 \cdot 4 = 8 > 4\delta(G) - 6$.

If $|V_{12}| = 2$, then $\delta(G) \leq d_G(v) \in \{1, 2\}$. Let $V_{12} = \{v_1, v_2\}$. Without loss of generality, we assume that $vv_1 \in E(G)$ when $d_G(v) = 1$, and $vv_1, vv_2 \in E(G)$ when $d_G(v) = 2$. If $d_G(v) = 1$, then by Proposition 3.1, $d_G(v_1) \geq 5$ and by Proposition 2.3(i), $d_G(v_2) \geq 2$; if $d_G(v) = 2$, then by Proposition 3.1, $d_G(v_i) \geq 4$ for $i = 1, 2$. So by (1), $|X| \geq (5-1)+(2-1) = 5 \geq 4\delta(G)-6$ ($\delta(G) = d_G(v) = 1$) or $|X| \geq (4-1) + (4-1) = 6 \geq 4\delta(G) - 6$ ($\delta(G) \leq d_G(v) = 2$).

If $|V_{12}| = 3$, then $\delta(G) \leq d_G(v) \in \{1, 2, 3\}$. Let $V_{12} = \{v_1, v_2, v_3\}$. Without loss of generality, we assume that $vv_1 \in E(G)$ when $d_G(v) = 1$, $vv_1, vv_2 \in E(G)$ when $d_G(v) = 2$, and $vv_1, vv_2, vv_3 \in E(G)$ when $d_G(v) = 3$. By a similar argument as above, if $d_G(v) = 1$, then $d_G(v_1) \geq 5$ and $d_G(v_i) \geq 2$ for $i = 2, 3$; if $d_G(v) = 2$, then $d_G(v_i) \geq 4$ for $i = 1, 2$ and $d_G(v_3) \geq 2$; if $d_G(v) = 3$, $d_G(v_i) \geq 3$ for $i = 1, 2, 3$. So by (1), $|X| \geq (5-1) + 2(2-1) = 6 \geq 4\delta(G) - 6$ ($\delta(G) \leq d_G(v) = 1$), or $|X| \geq 2(4-1) + (3-1) = 8 \geq 4\delta(G) - 6$ ($\delta(G) \leq d_G(v) = 2$), or $|X| \geq 3(3-1) = 6 \geq 4\delta(G) - 6$ ($\delta(G) \leq d_G(v) = 3$).

Thus Claim 3 is established and the lower bound can be obtained when $d_G(v_1) = d_G(v_2) = d_G(v_3) = d_G(v) = \delta(G) = 3$. □

Next we assume that V_{12} is a nontrivial vertex cut of G in Claims 4–6. Let G_1, G_2 be two nontrivial components of $G - V_{12}$. By Proposition 2.3(ii), we may assume that:

$$\text{every edge of the nontrivial component } G_i \text{ is colored by color } i \text{ for } 1 \leq i \leq 2. \tag{2}$$

Claim 4. *If $|V_{12}| = 3$ and V_{12} is a nontrivial vertex cut of G, then $|X| \geq 4\delta(G) - 6$. In particular,*

$$|X| \geq \begin{cases} 6 > 4\delta(G) - 6 \,:\, \text{if } 1 \leq \delta(G) \leq 2 \\ 8 > 4\delta(G) - 6 \,:\, \text{if } \delta(G) = 3 \\ \quad 4\delta(G) - 6 \,:\, \text{if } \delta(G) \geq 4. \end{cases}$$

Proof of Claim 4. Let $V_{12} = \{v_1, v_2, v_3\}$. First we prove the following fact.

Fact 1. There exists $v_i \in V_{12}$ such that $|E_1(v_i)| \geq 2$ and $|E_2(v_i)| \geq 2$, $i \in \{1, 2, 3\}$.

Proof of Fact 1. If not, then for each $i \in \{1, 2, 3\}$, there exists $j \in \{1, 2\}$ such that $|E_j(v_i)| = 1$. If $|E_1(v_1)| = |E_1(v_2)| = |E_1(v_3)| = 1$, then by Proposition 2.3(ii) and (2), $[V(G_1), V_{12}] \subseteq E_1(v_1) \cup E_1(v_2) \cup E_1(v_3)$ is an essential edge cut of size at most 3, contrary to $\kappa'_e(G) \geq 4$. Without loss of generality, we assume that $|[V(G_1), \{v_1\}]| \geq 2$ and $|[V(G_2), \{v_1\}]| = |[V(G_1), \{v_2\}]| = |[V(G_1), \{v_3\}]| = 1$. By $|[V(G_2), V_{12}]| \geq \kappa'_e(G) \geq 4$, without loss of generality, we may assume that $|[\{v_2\}, V(G_2)]| \geq 2$. Now if $v_1v_2 \in E(G)$, then by (2), v_1 satisfies Fact 1 if $f(v_1v_2) = 2$, and v_2 satisfies Fact 1 if $f(v_1v_2) = 1$. Thus, suppose that $v_1v_2 \notin E(G)$. Then $v_1v_3 \in E(G)$, otherwise $[V(G_2), \{v_1\}] \cup [V(G_1), \{v_2\}] \cup [V(G_1), \{v_3\}]$ is an essential edge cut of size at most 3, contrary to $\kappa'_e(G) \geq 4$. By (2), if $f(v_1v_3) = 2$ then v_1 satisfies Fact 1, so suppose that $f(v_1v_3) = 1$. Now if $|E_2(v_3)| \geq 2$, then v_3 satisfies Fact 1. So suppose that $|E_2(v_3)| = 1$. Recall that $v_1v_2 \notin E(G)$. Hence $v_2v_3 \in E(G)$, otherwise $[V(G_2), \{v_1\}] \cup [V(G_1), \{v_2\}] \cup [V(G_2), \{v_3\}]$ is an essential edge cut of size at most 3, contrary to $\kappa'_e(G) \geq 4$. By (2), if $f(v_2v_3) = 1$ then v_2 satisfies Fact 1, and if $f(v_2v_3) = 2$ then v_3 satisfies Fact 1.

By Fact 1, the first equality of (1) and Propositions 2.1(ii) and 2.3(i), we have that $|X| = \sum_{v_i \in V_{12}} |E_1(v_i)|\,|E_2(v_i)| \geq 2(\max\{4, \delta(G)\} - 2) + 2(\max\{2, \delta(G)\} - 1)$. Thus Claim 4 holds and the lower bound can be obtained when $d_G(v_1) = d_G(v_2) = d_G(v_3) = \delta(G) \geq 4$. □

Claim 5. *If $|V_{12}| = 2$ and V_{12} is a nontrivial vertex cut of G, then $|X| \geq 4\delta(G) - 10$. In particular,*

$$|X| \geq \begin{cases} \quad 8 > 4\delta(G) - 8 \,:\, \text{if } 1 \leq \delta(G) \leq 4 \\ \delta(G) + 8 \geq 4\delta(G) - 8 \,:\, \text{if } \delta(G) = 5 \\ \quad 4\delta(G) - 10 \,:\, \text{if } \delta(G) \geq 6. \end{cases}$$

Proof of Claim 5. Let $V_{12} = \{v_1, v_2\}$. We consider the following two cases:

Case 5.1. $|E_1(v_j)| \geq 2$ and $|E_2(v_j)| \geq 2$ for $1 \leq j \leq 2$.

Then $d_G(v_i) \geq 4$. By the case assumption, we have that $2 \leq |E_i(v_j)| \leq d_G(v_j)/2$ for $i, j \in \{1, 2\}$. By (1) and Proposition 2.1(ii), $|X| = \sum_{v_j \in V_{12}} |E_1(v_j)|\,|E_2(v_j)| \geq 2(d_G(v_1) - 2) + 2(d_G(v_2) - 2) \geq 4 \cdot \max\{\delta(G), 4\} - 8$. So

$$|X| \geq \begin{cases} 8 \geq 4\delta(G) - 8 \,:\, \text{if } 1 \leq \delta(G) \leq 4 \\ \quad 4\delta(G) - 8 \,:\, \text{if } \delta(G) \geq 5. \end{cases}$$

Case 5.2. Without loss of generality, we assume that $|E_1(v_1)| = 1$.

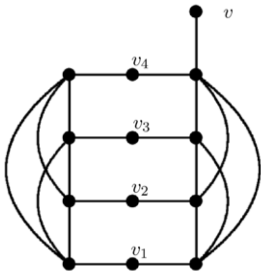

Fig. 1. $G_1, \delta(G_1) = 1$.

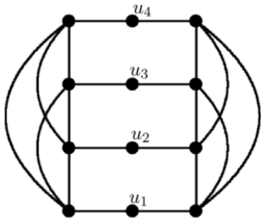

Fig. 2. $G_2, \delta(G_2) = 2$.

Since $[\{v_1\}, V(G_1)] \subseteq E_1(v_1)$, $|[\{v_1\}, V(G_1)]| \leq 1$. By Proposition 2.3(ii) and $|[V(G_1), V_{12}]| \geq \kappa'_e(G) \geq 4$, we have that

$$|E_1(v_2)| \geq |[\{v_2\}, V(G_1)]| \geq 3. \tag{3}$$

Next we show that

$$|E_2(v_2)| \geq 3. \tag{4}$$

If $v_1v_2 \notin E(G)$, then $|E_2(v_2)| \geq |[\{v_2\}, V(G_i)]| \geq 3$ (otherwise $[\{v_1\}, V(G_1)] \cup [\{v_2\}, V(G_2)]$ is an essential edge cut of size less than 4, a contradiction). So $v_1v_2 \in E(G)$ and $|[\{v_2\}, V(G_2)]| \geq 2$. Now if $f(v_1v_2) = 2$, then $|E_i(v_2)| = |[\{v_2\}, V(G_i)] \cup \{v_1v_2\}| \geq 3$ and so (4) holds. If $f(v_1v_2) = 1$, then $|[\{v_1\}, V(G_1)]| = 0$ by the assumption of Case 5.2. Thus $|E_2(v_2)| \geq |[\{v_2\}, V(G_2)]| \geq 3$ (otherwise $\{v_1v_2\} \cup [\{v_2\}, V(G_2)]$ is an essential edge cut of size less than 4, a contradiction). Hence (4) holds.

By (3) and (4), we have that $d_G(v_2) \geq 6$. By the first equality of (1) and Propositions 2.1(ii) and 2.3(i), we have that $|X| \geq (d_G(v_1) - 1) + 3(d_G(v_2) - 3) \geq \max\{2, \delta(G)\} - 1 + 3 \cdot \max\{6, \delta(G)\} - 9$. So

$$|X| \geq \begin{cases} 10 \ : \ \text{if } 1 \leq \delta(G) \leq 2 \\ \delta(G) + 8 \geq 6 \ : \ \text{if } \delta(G) = 3 \\ \delta(G) + 8 \geq 4\delta(G) - 8 \ : \ \text{if } 4 \leq \delta(G) \leq 5 \\ 4\delta(G) - 10 \ : \ \text{if } \delta(G) \geq 6. \end{cases}$$

Combining Cases 5.1 and 5.2, we establish Claim 5 and the lower bound can be obtained when $d_G(v_1) = d_G(v_2) = \delta(G) \geq 6$. □

Claim 6. *If $|V_{12}| = 1$ and V_{12} is a nontrivial vertex cut of G, then $|X| \geq 4\delta(G) - 16$. In particular,*

$$|X| \geq \begin{cases} 16 > 4\delta(G) - 16 \ : \ \textit{if } 1 \leq \delta(G) \leq 7 \\ 4\delta(G) - 16 \ : \ \textit{if } \delta(G) \geq 8. \end{cases}$$

Proof of Claim 6. Let $V_{12} = \{v_1\}$. Then for each $i = 1, 2$, $[V_{12}, V(G_i)]$ is an essential edge cut and so $|[V_{12}, V(G_i)]| \geq \kappa'_e(G) \geq 4$ for $i = 1, 2$. Thus $d_G(v_1) \geq 8$. By the equality of (1), Propositions 2.3(ii) and 2.1(ii), we have that $|X| \geq 4(d_G(v_1) - 4) \geq 4 \cdot \max\{\delta(G), 8\} - 16$. So Claim 6 holds and the lower bound can be obtained when $d_G(v_1) = \delta(G) \geq 8$. □

Note that every minimal essential edge cut of $L(G)$ corresponds to a minimal vertex cut of $L^2(G)$, and vice versa when $L^2(G)$ is not a complete graph. So it suffices to show that every essential edge cut of $L(G)$ has size at least equal to each bound in Theorems 1.3 and 1.4.

Proof of Theorem 1.3. By inspecting the bounds in Claims 1 through 6, and constructing the following graphs, we can see that Theorem 1.3 holds.

In Figs. 1 through 6, each graph G_i $(i = 1, 2, \ldots, 6)$ is a simple graph. In Figs. 4 through 6, H_j, M_j, J_j, $j = 1, 2$ represent subgraphs of G_i, $i = 4, 5, 6$ respectively.

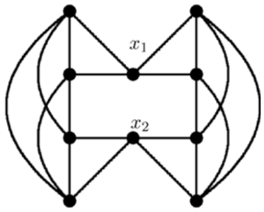

Fig. 3. G_3, $\delta(G_3) = 4$.

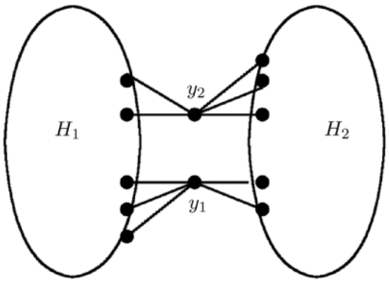

Fig. 4. G_4, $\delta(G_4) = 5$.

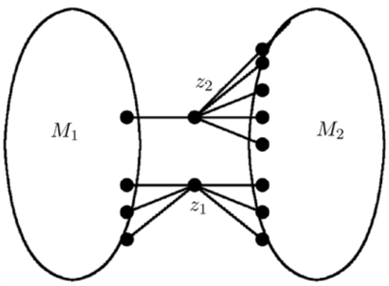

Fig. 5. G_5, $\delta(G_5) = 6$.

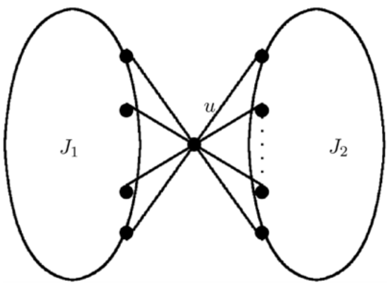

Fig. 6. G_6, $\delta(G_6) \geq 7$.

As shown in Fig. 1, we have that $d_{G_1}(v_i) = 2,\ i = 1, 2, 3, 4$, $d_{G_1}(v) = 1$ and $G_1 - \{v_1, v_2, v_3, v_4, v\}$ is isomorphic to two vertex disjoint K_4's; in Fig. 2, we have that $d_{G_2}(u_i) = 2,\ i = 1, 2, 3, 4$, and $G_2 - \{u_1, u_2, u_3, u_4\}$ is isomorphic to two vertex disjoint K_4's; in Fig. 3, we have that $d_{G_3}(x_i) = 4,\ i = 1, 2$, and $G_3 - \{x_1, x_2\}$ is isomorphic to two vertex disjoint K_4's; in Fig. 4, we have that $d_{G_4}(y_i) = 5,\ i = 1, 2$, $G_4 - \{y_1, y_2\}$ is the vertex disjoint union of H_1, H_2, each of which is isomorphic to K_5, and $d_{H_1}(y_1) = d_{H_2}(y_2) = 3$, $d_{H_2}(y_1) = d_{H_1}(y_2) = 2$; in Fig. 5, we have that $d_{G_5}(z_i) = 6$, $i = 1, 2$, $G_5 - \{z_1, z_2\}$ is the vertex disjoint union of M_1, M_2, each of which is isomorphic to K_8, and $d_{M_1}(z_1) = d_{M_2}(z_1) = 3$, $d_{M_1}(z_2) = 1$, $d_{M_2}(z_2) = 5$; in Fig. 6, we have that $d_{G_6}(u) \geq 8$, $G_6 - \{u\}$ is the vertex disjoint union of J_1, J_2, each of which is isomorphic to $K_{\delta(G_6)+1}$, and $d_{J_1}(u) = 4$, $d_{J_2}(u) = \max\{\delta(G_6), 8\} - 4$. Note that if $\delta(G_6) = 7$, then the vertex of minimum degree belongs to J_1 or J_2, and if $\delta(G_6) \geq 8$, then the vertex of minimum degree is u.

Each graph G_i ($i = 1, 2, \ldots, 6$) satisfies the condition $\kappa'_e(G_i) \geq 4$. And $\kappa'_e(L(G_1)) = \kappa'_e(L(G_2)) = 4$, $\kappa'_e(L(G_3)) = 8 = 4\delta(G_3) - 8$ where $\delta(G_3) = 4$, $\kappa'_e(L(G_4)) = 12 = 4\delta(G_4) - 8$ where $\delta(G_4) = 5$, $\kappa'_e(L(G_5)) = 14 = 4\delta(G_5) - 10$ where $\delta(G_5) = 6$, $\kappa'_e(L(G_6)) = 4\max\{\delta(G_6), 8\} - 16$ where $\delta(G_6) \geq 7$. Since $\kappa(L^2(G_i)) = \kappa'_e(L(G_i))$, each G_i ($i = 1, 2, \ldots, 6$)

obtains the bound for the minimum degree $1, 2, 4, 5, 6, 7$ respectively, and K_4 obtains the bound for the minimum degree 3. □

Proof of Theorem 1.4. Note that all possible cases are considered in Claims 1 through 6.

First we assume that $\kappa(G) \geq 3$. Since V_{12} is a vertex cut of size at most 2 of G in Claims 5 and 6, we only need to inspect Claims 1 through 4. In Claim 1, $|X| \geq 4\delta(G) - 4$, so $|X| \geq 4\delta(G) - 6$; in Claims 2 through 6, $|X| \geq 4\delta(G) - 6$. Thus the essential edge cut of $L(G)$ has a size of at least $4\delta(G) - 6$. Observe that $G = K_4$ obtains the bound $4\delta(G) - 6$.

If $\kappa(G) \geq 2$, by inspecting the bounds in Claims 1 through 5, we show that the essential edge cut of $L(G)$ has a size of at least $4\delta(G) - 10$. Observe that the graph G_5 depicted in Fig. 5 obtains the bound $4\delta(G) - 10$.

If $\kappa(G) \geq 1$, by inspecting the bounds in Claims 1 through 6, we show that the essential edge cut of $L(G)$ has a size of at least $4\delta(G) - 16$. Observe that the graph G_6 depicted in Fig. 6 obtains the bound $4\delta(G) - 16$. □

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